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# ON BEHAVIOR OF TWO-DIMENSIONAL NON-PLANE PARALLEL TURBULENT FLOWS OF INVISCID GAS*

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The time behavior of two-dimensional flows of inviscid gas in which the velocity component normal to the plane of independent variables and the vorticity components parallel to this plane are different from zero, is investigated. Equations of such flows form two different subsystems. The first subsystem describes a plane parallel ("primary") flow without the third velocity component, and is independent of the second subsystem consisting of a single equation for the third velocity component and determining the "secondary" flow. The flows are analyzed with sufficient detail without using numerical integration which carries with it unavoidable errors, and without linearization, both of which are employed to a lesser or greater degree in the study of the evolution of vortical structures (see /1—6/).

At the same time, the simplicity of the flows in question makes it easy to demonstrate the apparently very general, although not at all obvious properties of such a "determinate" system as the system of Euler equations. Such properties include the unlimited growth of vorticity, the appearance of tangential discontinuities not caused by the intersection of the shock waves, and "poor predictability" /4/. The above properties, which appear no matter how smooth the initial distributions, are connected with the kinematics of the fluid filaments.

Three types of primary flows are considered. Those depending on two coordinates $x$ and $y$ and on time $t$; the one-dimensional nonstationary flows depending on $x$ and $t$ but having two velocity components, and two-dimensional stationary flows depending on $x$ and $y$ only. The secondary flow (third velocity component) always depends on $x, y$ and $t$. In the first case an approach developed in /7/ is used to show the unlimited growth of vorticity along certain trajectories belonging to the flow boundary. For a plane parallel flow of an incompressible fluid an analogous result is established in /7/ for the gradient of the unique component of vorticity, and in the case of a three-dimensional flow for the vortex itself, although under the condition that the particle trajectory is rectilinear and belongs to the plane boundary. If the primary flow is one-dimensional or stationary, then the finite relationships obtained below describe the evolution of the vorticity (in this case its unlimited growth with $t\to\infty$) for the whole flow. An example showing the appearance of a tangential discontinuity is constructed for a stationary primary flow. In addition, a flow is constructed with arbitrarily smooth initial data, in which any three-dimensional moments (correlation functions) are different from zero only on the sets with zero measure.

1. Let $\mathbf{q}$ denote the velocity vector, $\rho$ the density and $p$ the pressure of the gas, and $\mathbf{F}$ the external mass force. Then the equation of motion of an inviscid compressible or incompressible medium will have the form

$$d\mathbf{q} / dt = -\rho^{-1}\nabla p + \mathbf{F} \quad (d / dt = \partial / \partial t + \mathbf{q}\nabla) \tag{1.1}$$

where $d/dt$ denotes the total derivative in $t$ along the trajectory of gas particles. If $\mathbf{F} = -\nabla U$, i.e. if the force is potential and the medium is two-parameter and, as in the case of thermodynamic equilibrium $Tds = di - (1/\rho)\,dp$ where $T$ is the absolute temperature, $s$ is the specific entropy and $i$ specific enthalpy, then (1.1) can be rewritten in the form

$$\partial\mathbf{q} / \partial t - \mathbf{q} \times \boldsymbol{\omega} = T\nabla s - \nabla I \tag{1.2}$$

$$(\boldsymbol{\omega} = \nabla \times \mathbf{q},\ 2I = 2i + q^2 + U,\ q = |\mathbf{q}|)$$

Taking the operation rot of (1.2), we arrive at the Helmholz equation in the form

$$d\boldsymbol{\omega} / dt = \nabla T \times \nabla s + (\boldsymbol{\omega}\nabla)\,\mathbf{q} - \boldsymbol{\omega}\nabla\mathbf{q} \tag{1.3}$$

The equations (1.1)—(1.3) hold even when no longer dependent on the concrete form of the energy equation. Moreover, even the passage to the equation

$$d\boldsymbol{\omega} / dt = (\boldsymbol{\omega}\nabla)\,\mathbf{q} - \boldsymbol{\omega}\nabla\mathbf{q} \tag{1.4}$$

*Prikl.Matem.Mekhan.,46,No.6,pp.972-978,1982 which from now on will replace (1.3), does not impose sufficiently rigid restrictions on the form of the energy equation. Indeed, replacing (1.3) by (1.4) is valid for any "barotropic" flows the realization of which by no means requires the necessary absence (together with the viscosity) of heat conductivity and other dissipative processes. Thus the equation (1.4) holds for the isothermal flows realized, in contrast, in the presence of an intense supply (or removal) of heat. For an incompressible fluid, (1.4) with $\nabla\mathbf{q} = 0$ represents the result of (1.1) in the presence of an external potential force. In the case of collisionless flows of perfect (inviscid and non-heat conducting) gas, their isentropic character and equation (1.4) are the natural consequences of the energy equation and the homogeneous initial fields of $s$.

We restrict the further analysis to the two-dimensional, non-plane parallel flows the all parameters of which are independent of the coordinate $z$ of the Cartesian $xyz$ -coordinate system. If $u$, $v$ and $w$ are the projections of $\mathbf{q}$ on its axes and $\omega_x$, $\omega_y$ and $\omega_z$ the analogous components of $\boldsymbol{\omega}$, then we have for such flows

$$\omega_x = \partial w / \partial y, \quad \omega_y = \partial w / \partial x, \quad \omega_z = \partial v / \partial x - \partial u / \partial y \tag{1.5}$$

Let $\mathbf{V}$ denote the projection of $\mathbf{q}$ on the $xy$ -plane. Since $w$ is independent of $z$, it follows that $\nabla\mathbf{q} = \nabla\mathbf{V}$ and the complete equation of continuity is reduce to the equation of continuity for the primary plane parallel flow, i.e.

$$d\rho / dt + \rho\nabla\mathbf{q} \equiv d\rho / dt + \rho\nabla\mathbf{V} = 0 \tag{1.6}$$

The passage from the first to the second version of (1.6) represents one of the prerequisites which make possible the decomposition of the flow into the primary and the secondary subflow. The previous premises come to the absence of $w$ in the energy equation (this is the case of the perfect gas) and to the independence on $w$ of the projections $F_x$ and $F_y$ of the force $\mathbf{F}$ on the axes $x$ and $y$. Finally, $\mathbf{F}$ is independent of $z$ in the flows considered, and the equation describing the secondary flow, i.e. the projection of (1.1) on the $z$-axis, has the form

$$dw / dt = F_z \tag{1.7}$$

When $F_z = 0$, (1.7) implies that $w$ is preserved in the particle.

2. First we shall see how $\boldsymbol{\omega}$ varies along the trajectories lying on the fixed impermeable boundaries. To do this we introduce, at every point of the boundary, a set of three unit vectors $\boldsymbol{\tau}, \mathbf{n}$, and $\mathbf{k}$ where $\mathbf{k}$ is collinear with the $z$-axis, $\tau$ is tangent to the boundary (the directions of $\tau$ and $\mathbf{V}$ coincide) and $\mathbf{n}$ is normal to the boundary. We denote the projections of $\boldsymbol{\omega}$ on $\tau$ and $\mathbf{n}$ by $\omega_\tau$ and $\omega_n$. The equation of continuity /1.6/ in the variables $\tau n$ assumes the form

$$\nabla\mathbf{q} \equiv \nabla\mathbf{V} \equiv \partial V / \partial\tau + V\partial\vartheta / \partial n = -\rho^{-1}d\rho / dt$$

where $V = |\mathbf{V}|$ and $\vartheta$ is the angle between $\mathbf{V}$ and the $x$-axis. Taking this into account, we obtain from (1.4) the following equations:

$$\begin{gathered} d\omega_\tau / dt = \omega_\tau\partial V / \partial\tau + (\omega_\tau / \rho)\, d\rho / dt + \omega_n\,(2VK - \omega_z) \\ d\omega_n / dt = -\omega_n\partial V / \partial\tau, \quad d\omega_z / dt = (\omega_z / \rho)\, d\rho / dt \end{gathered} \tag{2.1}$$

where $K = (\partial\vartheta / \partial\tau)_{n=0}$ is the curvature of the boundary. The last equation yields the known integral

$$\omega_z = \omega_{z0}\rho / \rho_0 \tag{2.2}$$

in which the index zero denotes the quantities at the point at which the particle was present at $t = 0$. According to (2.2) the vorticity of the primary flow is independent, as it should be, of the secondary flow. In contrast, the vorticity of the secondary flow depends on the parameters of the primary flow.

Let us combine the first equation of (2.1) multiplied by $\omega_n$, with the second equation multiplied by $\omega_\tau$, and then use the equation of continuity to eliminate $\nabla\mathbf{V} = \partial V / \partial\tau + V\partial\vartheta / \partial n$, while from the equation (2.2) to eliminate $\omega_z$. This yields

$$\frac{d}{dt}\left(\frac{\omega_n\omega_\tau}{\rho}\right) = \omega_n^2\chi \quad \left(\chi = 2\frac{VK}{\rho} - \frac{\omega_{z0}}{\rho_0}\right) \tag{2.3}$$

in which $\chi$ is defined in terms of the parameters of the primary flow. Let us assume that $\chi$ never vanishes along the trajectories under consideration. This will be true when $\omega_{z0} \neq 0$, e.g. on the plane boundaries where $K = 0$ and $\chi = \text{const}$, and not only on these boundaries. In such cases we can write (2.3) in the form

$$d\,(fg) / dt = f^2, \quad f = \omega_n\sqrt{|\chi|}, \quad g = (\omega_\tau \operatorname{sign}\chi) / (\rho\sqrt{|\chi|}) \tag{2.4}$$

The components $\omega_\tau$ and $\omega_n$, connected with $\omega_x$ and $\omega_y$ by the equations

$$\omega_\tau = \omega_x \cos\vartheta + \omega_y \sin\vartheta, \quad \omega_n = \omega_y \cos\vartheta - \omega_x \sin\vartheta$$

characterize, according to (1.5), the vorticity of the secondary flow. Since this flow does not affect the primary flow, it follows that the initial (at $t = 0$) distribution $w(x, y, 0) \equiv w_0(x, y)$, and as a result the values $\omega_{\tau 0}$ and $\omega_{n0}$ at the initial point of the trajectory do not affect $\chi$ and $\rho$ along this trajectory. Choosing the function $w_0(x, y)$ arbitrarily, we choose it such that the product $f_0 g_0 \equiv \omega_{\tau 0}\omega_{n0}(\mathrm{sign}\,\chi)/\rho_0$ is positive at the initial point. Then by virtue of the lemma proved in /7/, $f^2 + g^2$ and hence (for $0 < |\chi| < N < \infty$ which holds, as a rule) $\omega_\tau^2 + \omega_n^2$, grows indefinitely as $t \to \infty$. Since in the cases already considered the trajectory is curvilinear for $V \not\equiv \mathrm{const}$ even when $K = 0$, it follows that the situation described here differs from the example in /7/ where we have a rectilinear trajectory on a flat wall.

3. If the primary flow with $u$ and $v$ different from zero is one-dimensional, i.e. all its parameter are functions of $x$ and $t$ only, then the variation in $\boldsymbol{\omega}$ can be easily followed for all particles and not only for those moving along the impermeable boundaries. Indeed, let $w = w(x, y, t)$ and the remaining parameters be independent of $y$. By virtue of (1.1) this means, in particular, that $F_x$ and $F_y$ are independent of $y$ although $F_z$ may be a function of $y$ also. Restricting ourselves to the case of potential force we find, in accordance with (1.4) and (1.6), that for the one-dimensional primary flow

$$d\omega_x / dt = 0, \quad d\omega_y / dt = \omega_x\omega_z + (\omega_y / \rho)\, d\rho / dt$$

and $\omega_z$ satisfies (2.2). This implies that

$$\omega_x = \omega_{x0}, \quad \omega_y = (\omega_{y0} + \omega_{x0}\omega_{z0}t)\,\rho / \rho_0 \tag{3.1}$$

The first of the equations obtained becomes obvious by considering an elementary fluid area $S$ belonging, at $t = 0$, to some plane $x = x_0$. In the flow in question such an area element remains flat and retains the orientation of the normal, and its area, although it changes its form. This, and the invariance of the circulation along the contour $S$, together yield the first equation of (3.1). According to (3.1) and (2.2), $\omega_x$ and $\omega_z$ in the particle are bounded in the case of a one-dimensional primary flow, ( $\omega_z$ | increases proportionally to $\rho$ ) and $|\omega_y|$ increases with $t$ for $\omega_{x0}\omega_{z0} \neq 0$ almost linearly (with the accuracy of up to the variation in $\rho$).

4. Let us now assume that the primary flow is plane parallel and stationary. Then every of its streamlines can be regarded as an impermeable boundary for which the equations (2.1)—(2.4) hold and imply the unbounded growth of $|\boldsymbol{\omega}|$ with $t$ when $\omega_{\tau 0}\omega_{n0}\,\mathrm{sign}\,\chi > 0$. Moreover, in this case as in Sect. 3, we can derive explicit formulas which yield, together with (2.2), the description of the variation in all components of the vorticity along the particle trajectory. Indeed, by virtue of the stationary character of the primary flow $\partial V / \partial\tau = (dV / dt) / V$, and this enables us to integrate, first the second equation of (2.1), and then use this integral and (2.2) to integrate the first equation of (2.1). As a result we obtain

$$\omega_n = \omega_{n0}V_0 / V, \quad \omega_\tau = \left[\frac{\omega_{\tau 0}}{\rho_0 V_0} + \omega_{n0}V_0 \int_{\tau_0}^{\tau} \frac{\chi(\tau, \tau_0)}{V^3(\tau)}\, d\tau\right] \rho V \tag{4.1}$$

Here $\chi$ is the same as in (2.3), with its first term depending on $\tau$ in $\chi$ and the second term on $\tau_0$. The first equation of (4.1) can also be obtained by considering, as analogy in Sect. 3, the kinematic of the elementary fluid area element of the primary flow. Since the dependence of $t$ on $\tau$ is given in this case by the equation

$$t = \int_{\tau_0}^{\tau} \frac{d\tau}{V(\tau)} \tag{4.2}$$

and all parameters of the primary flow are known functions of $\tau$, it follows that (2.2), (4.1) and (4.2) together describe the evolution, in time, of all components of the vorticity. Thus $\omega_n$ increases without bounds only on the streamlines arriving at the critical points near which $V(\tau) \approx V_0 - \alpha(\tau - \tau_0)$ with the positive constant $\alpha$. According to (4.2) and (4.1) we have here $V \approx V_0 e^{-\alpha t}$, and

$$\omega_n \approx \omega_{n0} e^{\alpha t}, \quad \omega_\tau \approx -(\omega_{n0}\omega_{z0} / (2\alpha))\, e^{\alpha t} \tag{4.3}$$

The accuracy of (4.3) increases with increasing $t$ and with the approach of the initial point towards the critical point.

In the general case $\omega_\tau$ is a very complicated function of $t$. However if, as it happens in most cases, the integral in (4.1) diverges as $\tau \to \infty$, then $|\omega_\tau|$ grows without bounds as $t \to \infty$, although not necessarily monotonously. Here the sign constancy of $\chi$ is not required, although it formed a necessary element of the proof carried out in Sect.2.

As an example we shall show how $\omega_\tau$ varies along the trajectories which correspond to the closed streamlines without critical points. We can write (4.1) for them in the form

$$\omega_\tau(t) = \{F(n) + \Phi(t^\circ)\}R(t^\circ) \tag{4.4}$$

$$n = [t/T], \quad t^\circ = t - nT, \quad R = \rho V, \quad \Phi(t^\circ) = \omega_{n0}V_0 \int_0^{t^\circ} \frac{\chi(t^\circ)}{V^2(t^\circ)}\,dt^\circ$$

$$F(n) = \omega_{\tau 0}/R_0 + n\Phi(T)$$

Here $T$ is the period (time of a single circuit along the streamline), $[f]$ denotes the integral part of $f$, the relation connecting $\tau$ with $t$, following from (4.2), is assumed known (in (4.4) it is sufficient to know this connection for a single period). The curly brackets contain the sum of the step function $F$ and the discontinuous function $\Phi$ periodic in $t$. The discontinuities in $\Phi$ caused by the step-wise decrease in the value of $t^\circ$ from $T$ to 0 at the end of every period in $t$, are compensated by the jumps in $F$. As a result, the sum within the curly brackets yields a continuous function of $t$ which executes periodic oscillations about the straight line $\omega_\tau = F(t/T)$. The continous $t$-periodic function $R$ does not bring fundamental changes in the behavior of $\omega_\tau(t)$, and acts as a modulating multiplier. For the translational flow with shear, a flow induced in an unbounded space by a single vortex filament and for a Couette flow between coaxial rotating cylinders (as we know , this flow satisfies the Euler equations), the "primary" parameters on each streamline are constant. Here we have, by virtue of (4.1) and (4.2),

$$\omega_\tau(t) = \omega_{\tau 0} + \omega_{n0}(2VK - \omega_z)_0 t$$

with $\omega_z$ and $\omega_n$ unchanged in accordance with (2.2) and (4.1).

5. The unlimited growth of vorticity, i.e. of the derivatives of the velocity vector components, indicates the possibility of appearance of tangential discontinuities in the initially arbitrarily smooth liquid or gas flows, and demonstrates the "poor predictability" of the nonstationary vortical flows. Using the two-dimensional non-plane parallel flows we construct examples illustrating the properties in question more clearly than in Sect.2—4.

We begin the appearance of a tangential discontinuity. We take as the primary flow the plane parallel stationary attached flow past a sufficiently arbitrary nonsymmetric two-dimensional body immersed in a uniform incoming stream at $F = 0$. If the particles in such a flow were to pass from the leading stagnation point to the trailing stagnation point along the upper and lower generatrix in different, finite times $t_+$ and $t_-$, then the tangential discontinuity at $w_0(x, y) \neq \text{const}$ would appear behind the body after a finite period of time. This is obvious, since at $F = 0$, $w$ is retained by the particle according to (1.7), and with the times $t_\pm$ different and finite, the particles possessing different $w$ in the incoming flow would encounter each other behind the body after a finite time. The body can be replaced by a cylindrical "bubble". In the flow past this bubble, which could be symmetrical, the tangential discontinuity with finite $t_\pm$ would appear after a finite time at the boundary streamline of the bubble (the particles from the incoming flow do not penetrate the interior of the bubble).

The finite character of the times $t_\pm$ which played such an important part in the above discussion does not manifest itself in practice. Therefore at any $t < \infty$ the discontinuity $w$ is replaced in the above cases by a transitional layer of thickness $\delta$, the thickness rapidly decreasing to zero as $t \to \infty$. According to (4.2) we have in a flow pass a body or a stationary bubble $\delta \sim e^{-\alpha t}$, where the positive constants $\alpha$ determine, as in (4.3), the velocity field of the primary flow near the critical points.

Let us now construct a flow for which the spatial moments (correlation functions) characterizing in some sense the degree of order (or the lack of it) in the nonstationary flow can be calculated very simply. We take as the primary flow the stationary Couette flow between two coaxial cylinders of radius $r_-$ and $r_+$ respectively, with $0 \leqslant r_- < r_+ \leqslant \infty$. The streamlines is such a flow satisfying, as we said before, the Euler equations, are concentric circles, and when $\rho \equiv \text{const}$, the only velocity component (peripheral) is $V = V(r) = Ar + Br^{-1}$ where $r$ is the radial variable in the polar $r\varphi$ coordinates in the plane of flow with the origin on the axis of the cylinders, while $A$ and $B$ are constants related to the rates of rotation of the cylinders. In what follows, the only important aspects are the facts that $V$ is a function of $r$ only, and the angular velocity $\Omega \equiv V/r \not\equiv \text{const}$. Let $w_0(r, \varphi)$ be the initial distribution of $w$. Then from (1.7) with $F_z = 0$ we find that in the present case

$$w(r, \varphi, t) = w_0(r, \varphi - \Omega t) \tag{5.1}$$

Since $w_0$ is a $2\pi$-periodic function of the second argument, then by virtue of (5.1) the following formula holds for the mean value $\langle w \rangle = W$:

$$W(r,\varphi)=\lim_{t\to\infty}\frac{1}{t}\int\limits_0^t w(r,\varphi,t^\circ)\,dt^\circ=b_0(r)=\frac{1}{2\pi}\int\limits_0^{2\pi} w_0(r,\varphi)\,d\varphi$$

where $b_0$ is the zero coefficient of the expansion of $w_0(r,\varphi)$ into a Fourier series in $\varphi$. This, together with (5.1), yields the following expression for the nonstationary part of $w$, i.e. for the pulsation $w'\equiv w-W$:

$$w'(r,\varphi,t)=\sum_{n=1}^{\infty}\{a_n(r)\sin n(\varphi-\Omega t)+b_n(r)\cos n(\varphi-\Omega t)\} \tag{5.2}$$

$$a_n(r)=\frac{1}{2\pi}\int\limits_0^{2\pi} w_0(r,\varphi)\sin n\varphi\,d\varphi,\quad b_n(r)=\frac{1}{2\pi}\int\limits_0^{2\pi} w_0(r\,\varphi)\cos n\varphi\,d\varphi$$

Let us take in the $r\varphi$ -plane two points, $M$ and $M^\circ$, with coordinates $r,\varphi$ and $r^\circ,\varphi^\circ$ resepectively, and let us find a two-point moment (correlation function) of second order, depending also on the temporal shear $\tau$ and equal, by definition, to

$$B_{ww}(M,M^\circ,\tau)=\lim_{t\to\infty}\frac{1}{t}\int\limits_0^t w'(r,\varphi,t^\circ)\,w'(r^\circ,\varphi^\circ,t+\tau)\,dt$$

Substituting here the expressions (5.2) for the pulsations, multiplying and calculating the integral, we obtain

$$2B_{ww}=\Sigma\,[(a_na_k^\circ+b_nb_k^\circ)\cos\Phi^-+(a_nb_k^\circ-b_na_k^\circ)\sin\Phi^-]+ \tag{5.3}$$

$$\Sigma\,[(b_nb_k^\circ-a_na_k^\circ)\cos\Phi^++(a_nb_k^\circ+b_na_k^\circ)\sin\Phi^+]$$

$$(\Phi^\pm=n\varphi\pm k\varphi^\circ\mp k\Omega^\circ\tau)$$

Here $a_n, b_n, \Omega$ are the same functions of $r$ as in (5.2), while $a_k^\circ, b_k^\circ, \Omega^\circ$ are the same functions of $r^\circ$, and the sums are taken over the positive integers $n$ and $k$, which satisfy, for the given $r$ and $r^\circ$, the conditions

$$n\Omega(r)\mp k\Omega(r^\circ)=0 \tag{5.4}$$

where the minus (plus) sign corresponds to the first (second) sum in (5.3).

Let the point $M$ be fixed. Then $B_{ww}$ will be a function of $M^\circ$ and $\tau$ only. Moreover, since the summation in (5.3) is carried out not over all positive integral $n$ and $k$, but only over those satisfying (5.4), it follows that for a fixed $r$ the moment $B_{ww}=0$ almost everywhere in the annulus $r_-\leqslant r^\circ\leqslant r_+$. The exception is the set of null measure representing a collection of circles on which the ratio of the angular velocities $\Omega(r)/\Omega(r^\circ)=\pm n/k$ as well as the corresponding coefficents in the expansion (5.2), are not zero. If, e.g. the angular velocity $\Omega$ does not change its sign when $r_-\leqslant r\leqslant r_+$ and $w_0(r,\varphi)=b_0(r)+a_1(r)\sin\varphi+b_1(r)\cos\varphi$, then $B_{ww}\neq 0$ only on the circle $r^\circ=r$. Analogous results are obtained for the two-point, three-point, etc. moments of not only the second, but also of the higher orders.

We emphasise that the latter by no means implies the chaotic (turbulent) character of the flow in question, and even of the fields $w'$. In fact, the primary flow in the above example is laminar in the usual sense, the dependence of $w'$ on the coordinates and time is by virtue of (5.1) fully determined, and the trajectory of each particle is a spiral line. We note that if the primary flow is stationary, here as well as in Sect.4, then the complete and the linearized (with reference to the stationary flow) equations for $w$, $\omega_x$ and $\omega_y$ or for $\omega_\tau$ and $\omega_n$ do not differ from each other, and $w$ can always be regarded as a "label" which identifies the particles like a spot of dye. The only difference lies in the fact that the dye concentration must be sufficiently low so as not to affect the density of the medium, while the quantity $w$ in all flows in question satisfying the complete Euler equations can be of arbitrary magnitude.

Notwithstanding the latter remarks, the results obtained, and above all, the almost identical equality to zero of the correlation functions demonstrate without any doubt the complexity of the motion of the liquid particles. Even greater complexity will be observed if the plane parallel vortical nonstationary flow, induced e.g. by several vortex filaments, is taken as the primary flow (the filaments themselves are very complicated /1,2/) although even here the secondary flow does not affect the primary flow. It is therefore difficult even to imagine how great a chaos will result in the analogous situation for the general, three-dimensional case when nonlinear effects caused by mutual twisting of vortex tubes about each other begin to manifest themselves /8/. Another sourse of the chaotic behavior will, in any case, be the instability of the resulting flows with respect to the small, uncontrolled perturbations, their influence increasing, as a rule, with the evolution of the flow. Indeed, even in the last example when the initial profile of $w_0$ over $r$ is arbitrary, an arbitrarily large number of the points of inflection form with increasing $t$.

The viscosity, no matter how small, exerts a smoothing effect /9,10/ by interfering with the unlimited growth of the derivatives. We shall show how it happens in a gap of height $h = r_+ - r_-$ between two rotating cylinders at $\mathrm{Re}_1 \equiv \Omega_+ r_+^2/\nu \gg 1$ and $\mathrm{Re}_2 \equiv \Omega_+ h^2/\nu \gg 1$ where $\nu$ is the kinematic viscosity. Analysis of the development of such a flow within the framework of the Navier—Stokes equations shows that at first the influence of the viscosity is practically nil. Later its effect manifests itself in two stages. In the first stage the distribution of $w$ becomes axisymmetric with $w \simeq W(r)$ everywhere outside the thin laminary layer next to the walls, i.e. with the $z$-component of the momentum of each cylindrical layer $r = \mathrm{const}$ preserved. The controlling parameter at this stage is the time of evolution of the first harmonic of the expansion (5.2)

$$t_1 = (3/\nu)^{1/3}(d\Omega/dr)_+^{-2/3}$$

The peripheral nonuniformity vanishes completely when $(t/t_1)^3 \gg 1$, although the higher harmonics decay more rapidly already when $(t/t_1)^3 \gg n^{-2}$. When $(t/t_1)^3 \ll n^{-2}$, the viscosity outside the boundary layers does not affect the harmonics up to and including the $n$-th order. The viscosity begins to affect the axisymmetric flow with $w \simeq W(r) \neq 0$ strongly, at the time commensurable with $t_2 = h^2/(\nu\lambda^2)$ where $\lambda$ is a constant of the order of unity ($\lambda \simeq 2.4$ for $h/r_+ = 1$ and $\lambda = \pi$ for $h/r_+ = 0$). When $t/t_2 \gg 1$, the velocity component $w \ll W(r)$. Since

$$t_2/t_1 = h^2(d\Omega/dr)_+^{2/3}\nu^{-2/3}\lambda^{-2}3^{-1/3} \sim (\mathrm{Re}_2)^{2/3}$$

it follows that at $\mathrm{Re}_2 \gg 1$ the flow, after smoothing the peripheral inhomogeneity, is practically unaffected by the viscosity for a very long time over almost the whole gap.

The mechanism of the decay of $w$ described above presupposes the stability of the flow. If the solution (5.1) is found to be unstable already at $t < t_1$ (which is perfectly possible) then its evolution will be different.

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